Week 1: Probabilistic Inequalities

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1 Concentration Inequalities

Theorem 1.1. *(Markov' inequality, [Dur96]) Let X be a nonnegative random variable, then we have $\mathsf{P}[X \geq t\mathsf{E}[X]] \leq \frac{1}{t}$ for any $t > 0$.*

Proof. By definition, we have

$$\begin{aligned}\mathsf{E}[X] = \int_0^\infty xp(x)dx &= \int_0^{t\mathsf{E}[X]} xp(x)dx + \int_{t\mathsf{E}[X]}^\infty xp(x)dx \\ &\geq \int_0^{t\mathsf{E}[X]} 0p(x)dx + \int_{t\mathsf{E}[X]}^\infty t\mathsf{E}[X]p(x)dx \\ &= t\mathsf{E}[X]\int_{t\mathsf{E}[X]}^\infty p(x)dx := t\mathsf{E}[X]\mathsf{P}[X \geq t\mathsf{E}[X]].\end{aligned}$$

Dividing both side by $t\mathsf{E}[X]$, then we arrive at $\mathsf{P}[X \geq t\mathsf{E}[X]] \leq \frac{1}{t}$. □

Theorem 1.2. *(Chebyshev's inequality, [Dur96]) Let X be a random variable, then we have $\mathsf{P}[|X - \mathsf{E}[X]| \geq t\sqrt{\mathsf{Var}[X]}] \leq \frac{1}{t^2}$ for any $t > 0$.*

Proof. Define a new random variable $Y = (X - \mathsf{E}[X])^2$, thus $E[Y] = \mathsf{Var}[X]$. By Markov' inequality, we have

$$\mathsf{P}[Y \geq t^2 E[Y]] \leq \frac{1}{t^2}.$$

Taking square root of Y, then we arrive at $\mathsf{P}[|X - \mathsf{E}[X]| \geq t\sqrt{\mathsf{Var}[X]}] \leq \frac{1}{t^2}$. □

Corollary 1.3. *(One side Chebyshev's inequality) Let X be a random variable, then we have $\mathsf{P}[X \geq \mathsf{E}[X] + t\sqrt{\mathsf{Var}[X]}] \leq \frac{1}{1+t^2}$ for any $t > 0$.*

2 Convergence to Gaussian

A random variable with Gaussian distribution or normal distribution $N(0,1)$ has probability density function as

$$p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Inductively, the probability density function of a multivariate Gaussian distribution $N(0, I_n)$ (I_n is an $n \times n$ identity matrix) is defined as

$$p(x) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{\|x\|_2^2}{2}}.$$

There are three properties regarding Gaussian random variables:

Property 1. $p(x)$ is rotationally invariant;

Property 2. $p(x)$ can be viewed as a product of n single-variate Gaussian distribution probability density functions ;

Property 3. Consider two independent Gaussian random variable $X_1 \sim N(0,1), X_2 \sim N(0,1)$, then $ax_1 + bx_2 \sim N(0, a^2 + b^2)$.

Theorem 2.1. *(Central Limit Theorem, [Dur96]) Let $X_1, \ldots, X_n$ be i.i.d. random variables whose summation is Y_n (i.e., $Y_n = \sum_{i=1}^n X_i$), then if $var[X_1] \in (0, \infty)$, $\frac{Y_n}{n}$ converges in distribution to $N(\mathsf{E}[X_1], \frac{1}{n}\mathsf{Var}[X_1])$.*

Theorem 2.2. *(Berry-Esseen Theorem, [Ber41]) Let $X_1, \ldots, X_n$ be independent random variables whose summation is Y_n (i.e., $Y_n = \sum_{i=1}^n X_i$) and $Z_n \sim N(\mathsf{E}[Y_n], \mathsf{Var}[Y_n])$, then for any t,*

$$|\mathsf{P}(Y_n \leq t) - \mathsf{P}(Z_n \leq t)| \leq C \frac{\sum_{i=1}^n \mathsf{E}[|X_i|^3]}{[\mathsf{Var}[Y_n]]^{3/2}}, \tag{1}$$

where $C \in (0,1)$ is a constant.

Please note that in (1), we can replace $\leq$ by $>$ since $\mathsf{P}(Y_n \leq t) = 1 - \mathsf{P}(Y_n > t)$ and $\mathsf{P}(Z_n \leq t) = 1 - \mathsf{P}(Z_n > t)$.

Example 2.3. Consider $X_1, \ldots, X_n$ be i.i.d. random variables where for each i,

$$X_i = \begin{cases} 1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases},$$

then we have $\mathsf{E}[Y_n] = 0, \mathsf{Var}[Y_n] = n$. Thus, (1) yields

$$|\mathsf{P}(Y_n \geq t) - \mathsf{P}(Z_n \geq t)| \leq C \frac{\sum_{i=1}^n \mathsf{E}[|X_i|^3]}{[\mathsf{Var}[Y_n]]^{3/2}} := \frac{Cn}{n^{3/2}} = \frac{C}{n^{1/2}}$$

which will go to zero if $n \to \infty$. ♦

2.1 Bounding normal distribution

Let $Z \sim N(0,1)$. Let us bound the error function of normal distribution defined as for each $t > 0$

$$\phi(t) := \mathsf{P}(Z > t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{x^2}{2}} dx.$$

The next lemma shows $\phi(t)$ can be tightly approximated.

Lemma 2.4. $\phi_{LB}(t) \leq \phi(t) \leq \phi_{UB}(t)$ *for any* $t > 0$, *where* $\phi_{UB}(t) := \frac{1}{\sqrt{2\pi}}\frac{1}{t}\left(1-\frac{1}{t^2+1}\right)e^{-\frac{t^2}{2}}, \phi_{UB}(t) := \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t}$

Proof.

(1) First of all, to upper bound $\phi(t)$, note that for any $x \geq t$, we have $x/t \geq 1$, thus

$$\phi(t) \leq \frac{1}{\sqrt{2\pi}}\int_t^\infty \frac{x}{t}e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t} = \phi_{UB}(t).$$

(2) Now we claim that $\phi_{LB}(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{t}\left(1-\frac{1}{t^2+1}\right)e^{-\frac{t^2}{2}}$ is a lower bound of $\phi(t)$. To prove it, let us define a new function

$$g(t) = \phi(t) - \phi_{LB}(t) = \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-\frac{x^2}{2}}dx - \frac{1}{\sqrt{2\pi}}\frac{1}{t}\left(1-\frac{1}{t^2+1}\right)e^{-\frac{t^2}{2}},$$

whose first derivative is

$$g'(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\left[-1+\frac{t^2-1}{(t^2+1)^2}+\frac{t^2}{t^2+1}\right] = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\frac{-2}{(t^2+1)^2} < 0.$$

$g(t)$ is monotone decreasing, thus $g(t) \geq \lim_{x\to\infty} g(x) = 0$, i.e. $\phi(t) \geq \phi_{LB}(t)$.

Thus, $\phi(t) \approx \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t}$.

□

In Example 2.3, we have

$$\mathsf{P}(Y_n \geq t\sqrt{n}) \approx \mathsf{P}(Z_n \geq t\sqrt{n}) \leq \frac{1}{t\sqrt{2\pi}}e^{-\frac{t^2}{2}}$$

which would go to zero if $t = \omega(n)$ such that $\lim_{n\to\infty}\omega(n) = \infty$.

3 Routing with minimum congestion

Given a non-directed graph $G(V,E)$, let us define Congestion((i,j)) as number of paths using edge $(i,j) \in E$. Then in this problem, we are trying to find k paths $(s_1,t_1),\ldots,(s_k,t_k)$ with $s_i,t_i \in V$ for all i, such that the largest congestion across all the edges are minimized. It turns out that such problem is NP-hard even with $k = 2$.

We now consider relaxing the path into flow (i.e., we are sending one unit of flow from s_i to t_i for each i), which is polynomially solvable. The following example illustrates how the relaxation works.

Example 3.1. Consider a graph with four nodes (s_1,s_2,t_1,t_2) and four edges $(s_1,t_2),(t_1,t_2),(s_2,t_1),(s_1,s_2)$ (see Figure 1). In this example, if we solve the original problem the optimal value is 2, because the paths of (s_1,t_1) and (s_2,t_2) share at least one edge. However, if we solve the relaxed one, the optimal value is 1 since for (s_1,t_1) flow, we can send half flow through $s_1-t_2-t_1$ and another half flow through $s_1-s_2-t_1$, and the same strategy applies to (s_2,t_2) flow. ♦

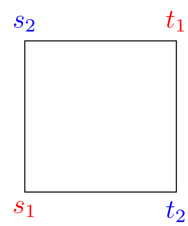

FIGURE 1: Illustration of a four-node example

Now, for each $l \in \{1, \ldots, k\}$, let us define a set of flow variables $\{x_{ij}^l\}_{(i,j)\in E}$ which forms a flow with value 1 between s_l and t_l. Clearly, Congestion((i, j)) can be expressed as a function of $\{x_{ij}^l\}_{(i,j)\in E}$, i.e., Congestion($(i, j)$) $= \sum_{l=1}^k x_{ij}^l$. Denote α to be the maximum congestion across the edges. Thus, our relaxed problem is formulated as follows:

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} & \sum_{l=1}^{k} x_{ij}^l \le \alpha, \forall (i,j) \in E \\ & \{x_{ij}^l\}_{(i,j)\in E} \text{ forms a flow with value 1 between } s_l \text{ and } t_l \\ & 0 \le x_{ij}^l \le 1, \forall (i,j) \in E, l \in l \in \{1, \ldots, k\} \end{aligned}$$

which can be solved as a linear program (LP). Let $\{x_{ij}^{l*}\}_{(i,j)\in E}, \alpha^*$ be the optimal solution.

Now we are ready to prove main result in this section.

Theorem 3.2. *With probability at least* $\frac{1}{2}$*, there exists a routing in* $G(V, E)$ *whose maximum congestion is no larger than* $\max((2e-1)\alpha^*, 2\ln n) \le (2e-1)\alpha^* + 2\ln n$.

Proof.

(1) First of all, for each $l \in \{1, \ldots, k\}$, we can decompose the optimal flow between s_l and t_l by path. This can be done iteratively: at iteration τ, suppose we find a path $P_\tau(s_l, t_l)$ from s_l to t_l, let f_τ^l be the minimum edge flow on this path, then reduce the flow of each edge on this path by f_τ^l; go to next iteration until we cannot find any path with positive flow. This procedure takes at most $|E|$ iterations as at each iteration at least one edge is removed. Hence, after flow decomposition, we arrive at the following equality

$$\text{flow}(s_l, t_l) = \sum_\tau f_\tau^l P_\tau(s_l, t_l), \forall l \in \{1, \ldots, k\}$$

and $\sum_\tau f_\tau^l = 1$ with $f_\tau^l \in [0, 1]$ for all τ. We see that for each $l \in \{1, \ldots, k\}$, $\{f_\tau^l\}_\tau$ can be viewed as a probability distribution of paths $\{P_\tau(s_l, t_l)\}_\tau$. This inspires us to define independent random variables $\{Y_l\}$ for each edge (i, j) as

$$Y_l = \begin{cases} 1, & \text{if } (i,j) \text{ is used for } (s_l, t_l) \text{ with probability } p_l \\ 0, & \text{otherwise} \end{cases}.$$

Then, Congestion((i, j)) $:= Y = \sum_{l=1}^k Y_l$ and let

$$\mu := \mathsf{E}[Y] = \sum_{l=1}^{k} \mathsf{E}[Y_l] = \sum_{l=1}^{k} p_l := \sum_{l=1}^{k} x_{ij}^{l*} \le \alpha^*.$$

(2) Now we would like to bound Y. Given $\delta, t > 0$, then we have

$$
\begin{aligned}
\mathsf{P}[Y \geq (1+\delta)\mu] &= \mathsf{P}[e^{tY} \geq e^{t(1+\delta)\mu}] && (e^x \text{ is monotone increasing})\\
&\leq \frac{\mathsf{E}[e^{tY}]}{e^{t(1+\delta)\mu}} && (\text{Markov's inequality})\\
&= \frac{\mathsf{E}[e^{t\sum_{l=1}^k Y_l}]}{e^{t(1+\delta)\mu}} && (Y = \sum_{l=1}^k Y_l)\\
&= \frac{\prod_{l=1}^k \mathsf{E}[e^{tY_l}]}{e^{t(1+\delta)\mu}} && (\text{independence of } \{Y_l\})\\
&= \frac{\prod_{l=1}^k [(1-p_l)+p_l e^t]}{e^{t(1+\delta)\mu}} = \frac{\prod_{l=1}^k [1+p_l(e^t-1)]}{e^{t(1+\delta)\mu}} && (\text{definition of } \{Y_l\})\\
&\leq \frac{\prod_{l=1}^k e^{p_l(e^t-1)}}{e^{t(1+\delta)\mu}} && (1+p_l(e^t-1) \leq e^{p_l(e^t-1)})\\
&= \left[\frac{e^{e^t-1}}{e^{t(1+\delta)}}\right]^\mu && (\mu = \sum_{l=1}^k p_l).
\end{aligned}
$$

Note that by minimizing $e^t - 1 - t(1+\delta)$ over $t > 0$, the minimizer is $t^* := \ln(1+\delta)$. Substitute $t = \ln(1+\delta)$, the above inequality yields

$$
\begin{aligned}
\mathsf{P}[Y \geq (1+\delta)\mu] &\leq \left[\frac{e^{e^{\ln(1+\delta)}-1}}{e^{(1+\delta)\ln(1+\delta)}}\right]^\mu = \left[\frac{e^\delta}{(1+\delta)^{1+\delta}}\right]^\mu && (t = \ln(1+\delta))\\
&\leq \frac{1}{2^{-(1+\delta)\mu}} && (\text{if } \delta \geq 2e-1)\\
&\leq \frac{1}{n^2} && (\text{if } (1+\delta)\mu \geq 2\ln n).
\end{aligned}
$$

As there are at most $\frac{n^2}{2}$ edges, thus according to union bound, the probability that the maximum congestion is no larger than $\max((2e-1)\alpha^*, 2\ln n) \leq (2e-1)\alpha^* + 2\ln n$ is at least $\frac{1}{2}$.

□

This example arises an important concentration bound.

Theorem 3.3. *(Chernoff bound, [Che52]) Let $X_1, \ldots, X_n \in [0,1]$ be independent random variables, and $X := \sum_{i=1}^n X_i$, then we have $\mathsf{P}[X > (1+\delta)\mathsf{E}[X]] \leq e^{-\frac{\delta^2}{2+\delta}\mathsf{E}[X]}$ for any $\delta > 0$.*

Moreover, if $\delta \in (0,1)$, we have $\mathsf{P}[X < (1-\delta)\mathsf{E}[X]] \leq e^{-\frac{\delta^2}{2}\mathsf{E}[X]}$

A more general inequality is

Theorem 3.4. *(Hoeffding inequality, [Hoe63]) Let $X_1, \ldots, X_n$ be independent random variables with $X_i \in [a_i, b_i]$, and $X := \sum_{i=1}^n X_i$, then we have $\mathsf{P}[X > \mathsf{E}[X] + t] \leq e^{-\frac{2t^2}{\sum_{i=1}^n (a_i-b_i)^2}}$ and $\mathsf{P}[X < \mathsf{E}[X] - t] \leq e^{-\frac{2t^2}{\sum_{i=1}^n (a_i-b_i)^2}}$ for any $t > 0$.*

4 Johnson-Lindenstrauss Lemma

In this section, let us consider m vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ and we would like to demonstrate that for any $\epsilon \in (0,1)$, there exists an integer k and $v_1, \ldots, v_m \in \mathbb{R}^k$ such that $(1-\epsilon)\|u_i - u_j\|_2^2 \leq \|v_i - v_j\|_2^2 \leq (1+\epsilon)\|u_i - u_j\|_2^2$ for all i, j. Johnson-Lindenstrauss Lemma tells us that $k = O(\log(m)/\epsilon^2)$ suffices.

First of all, let us define random projection as follows. Let $R \in \mathbb{R}^{n \times k}$ be an entry-wise independent Gaussian random matrix, where $R_{ij} \sim N(0,1)$, then the linear projection of u_i is defined as $v_i = \frac{1}{\sqrt{k}} R^\top u_i$ for each i. Now we are ready to show Johnson-Lindenstrauss Lemma.

Theorem 4.1. *(Johnson-Lindenstrauss Lemma, [JL84]) Given $\epsilon \in (0,1)$, let $u \in \mathbb{R}^n$ and $v = \frac{1}{\sqrt{k}} R^\top u$, then*

$$\mathsf{P}[|\|u\|_2^2 - \|v\|_2^2| > \epsilon \|u\|_2^2] \leq 2e^{-\frac{k}{4}(\epsilon^2 - 2\epsilon^3/3)}. \tag{2}$$

In particular, we choose $k = O(\log(m)/\epsilon^2)$, then with probability at least $\frac{9}{10}$, $(1-\epsilon)\|u_i - u_j\|_2^2 \leq \|v_i - v_j\|_2^2 \leq (1+\epsilon)\|u_i - u_j\|_2^2$ for all i, j.

Proof. First of, if $u = 0$, then (2) holds trivially. Now we assume that $u \neq 0$, thus without loss of generality, we can normalize it as $\|u\|_2 = 1$.

Note that

$$k\|v\|_2^2 = \sum_{l=1}^{k} (R_{\cdot l}^\top u)^2 := \sum_{l=1}^{k} Y_i^2 = Y.$$

We also have

$$\mathsf{E}[Y] = k\mathsf{E}[\|v\|_2^2] = \mathsf{E}[u^\top R R^\top u] = u^\top \mathsf{E}[R R^\top] u = k$$

where the last equality is due to $\|u\|_2^2 = 1$ and

$$\mathsf{E}[RR^\top]_{ij} = \begin{cases} \sum_{l=1}^k \mathsf{E}[R_{il}R_{jl}] = \sum_{l=1}^k \mathsf{E}[R_{il}^2] = k, & \text{if } i = j \\ \sum_{l=1}^k \mathsf{E}[R_{il}R_{jl}] = \sum_{l=1}^k \mathsf{E}[R_{il}]\mathsf{E}[R_{jl}] = 0, & \text{otherwise} \end{cases}.$$

Thus, (2) is equivalent to show

$$\mathsf{P}[|Y - \mathsf{E}[Y]| > \epsilon \mathsf{E}[Y]] \leq 2e^{-\frac{k}{4}(\epsilon^2 - 2\epsilon^3/3)}. \tag{3}$$

Indeed, if (3) holds, then choose k such that $2e^{\left(-\epsilon^2 + \epsilon^3/3\right)\frac{k}{4}} < \frac{1}{10m^2}$, i.e. $k = O(\log(m)/\epsilon^2)$, and according to union bound, the probability that $(1-\epsilon)\|u_i - u_j\|_2^2 \leq \|v_i - v_j\|_2^2 \leq (1+\epsilon)\|u_i - u_j\|_2^2$ for all i, j is at least $\frac{9}{10}$.

We will prove (3) in the next lemma. □

Lemma 4.2. *Suppose $Y_1, \ldots, Y_k$ are i.i.d. $N(0,1)$ random variables and $Y := \sum_{l=1}^k Y_i^2$, then* (3) *holds.*

Proof. Note that

$$\mathsf{P}[|Y-\mathsf{E}[Y]|>\epsilon\mathsf{E}[Y]]\le\mathsf{P}[Y>(1+\epsilon)\mathsf{E}[Y]]+\mathsf{P}[Y<(1-\epsilon)\mathsf{E}[Y]].$$

We will first bound $\mathsf{P}[Y>(1+\epsilon)\mathsf{E}[Y]]$. Indeed, for any $t>0$,

$$\begin{aligned}
\mathsf{P}[Y>(1+\epsilon)\mathsf{E}[Y]] &= \mathsf{P}[e^{tY}>e^{t(1+\epsilon)k}] && (\mathsf{E}[Y]=k)\\
&\le \frac{\mathsf{E}[e^{tY}]}{e^{t(1+\epsilon)k}} && \text{(Markov's inequality)}\\
&= \prod_{l=1}^{k}\frac{\mathsf{E}[e^{tY_i^2}]}{e^{t(1+\epsilon)k}} && (Y=\sum_{l=1}^{k}Y_i^2 \text{ and by independence of } \{Y_i\})\\
&= \left[\frac{1}{\sqrt{1-2t}e^{t(1+\epsilon)}}\right]^k && (Y_i\sim N(0,1) \text{ and } \mathsf{E}[e^{tY_i^2}]=\frac{1}{\sqrt{1-2t}} \text{ by Claim 4.3})\\
&= \left[\frac{1}{(1-2t)e^{2t(1+\epsilon)}}\right]^{\frac{k}{2}}
\end{aligned}$$

Claim 4.3. *Let $X\sim N(0,1)$, then $\mathsf{E}[e^{tX^2}]=\frac{1}{\sqrt{1-2t}}$ for any $t\in(0,\frac{1}{2})$.*

Proof. It can be shown by direct calculation

$$\begin{aligned}
\mathsf{E}[e^{tX^2}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx^2}e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{1-2t}}\left[\frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{1-2t}}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2\frac{1}{1-2t}}}dx\right]\\
&= \frac{1}{\sqrt{1-2t}}
\end{aligned}$$

where the last inequality is because $\frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{1-2t}}}e^{-\frac{x^2}{2\frac{1}{1-2t}}}$ is the density function of Gaussian random variable $N(0,\frac{1}{1-2t})$ □

We would like to maximize $(1-2t)e^{2t(1+\epsilon)}$ over $t>0$, whose maximizer is $t^*=\frac{\epsilon}{2(1+\epsilon)}$. Thus, $\mathsf{P}[Y>(1+\epsilon)\mathsf{E}[Y]]$ can be further bounded as

$$\begin{aligned}
\mathsf{P}[Y>(1+\epsilon)\mathsf{E}[Y]] &\le \left[\frac{1}{(1-2t)e^{2t(1+\epsilon)}}\right]^{\frac{k}{2}}=\left[\frac{1+\epsilon}{e^{\epsilon}}\right]^{\frac{k}{2}} && (\text{set } t=\frac{\epsilon}{2(1+\epsilon)})\\
&\le e^{\left(-\frac{\epsilon}{2}+\frac{\epsilon^3}{3}\right)\frac{k}{2}} && (\text{set } 1+\epsilon\le e^{\epsilon-\frac{\epsilon^2}{2}+\frac{\epsilon^3}{3}},\forall\epsilon\in(0,1)).
\end{aligned}$$

By the similar reasoning, we can also bound

$$\mathsf{P}[Y<(1-\epsilon)\mathsf{E}[Y]]\le e^{\left(-\epsilon^2+2\epsilon^3/3\right)\frac{k}{4}}.$$

Hence, (3) holds.

□

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