Random Graphs

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1 Random Graphs

An Erdős-Renyi random graph $G_{n,p}$ is defined as a graph on n vertices where each edge (out of the possible $\binom{n}{2}$ edges) is in the graph independently with probability p. This is a simple model for generating a graph randomly. Even though this model may not resemble real-life examples, it can be useful for analyzing computational complexity. Random graphs can indicate an type of average case performance for algorithms of this particular distribution of graphs.

A monotone property P of a $G_{n,p}$ is one such that if P holds with asymptotically almost surely for G_{n,p_1}, then P also holds a.a.s. for G_{n,p_2}. One example of a monotone property is the property of having a Hamiltonian cycle.

A value p^* is a sharp threshold for a graph property P if for any $\epsilon > 0$,

$$\mathsf{P}(P) \to \begin{cases} 0 & \text{if } \frac{p}{p^*} < 1 - \epsilon \\ 1 & \text{if } \frac{p}{p^*} > 1 + \epsilon \end{cases} \tag{1}$$

Theorem 1.1. *Every monotone graph property has a sharp threshold.*

We will not prove this sharp threshold theorem, but we will prove that graph connectivity in an Erdős-Renyi random graph has a type of threshold in the following sense:

Theorem 1.2. (Connectivity threshold). *Let $p = \frac{\ln n + c}{n}$. Then,*

$$\mathsf{P}(G_{n,p} \text{ is connected}) \to \begin{cases} 0 & \text{if } c \to -\infty \\ e^{-e^{-c}} & \text{for } c \text{ fixed w.r.t. } n \\ 1 & \text{if } c \to \infty \end{cases} \tag{2}$$

Proof. We will prove that $\mathsf{P}(G_{n,p}$ is connected$) \to 1$ if $c \to \infty$. To begin, we will use the following:

$$\begin{aligned} \mathsf{P}(\exists \text{isolated vertex}) &\le \mathsf{P}(G_{n,p} \text{ is not connected}) && (3) \\ &= \mathsf{P}(\exists \text{isolated component of size } k, 1 \le k \le n/2) && (4) \\ &\le \mathsf{P}(\exists \text{isolated vertex}) + o(1) && (5) \end{aligned}$$

The first inequality follows because if G has an isolated vertex, G must be disconnected. Clearly, a graph is connected iff it has no isolated components of size less than n, which occurs iff there are no isolated components of size less than $n/2$. We will now prove the inequality in the third line.

Let X_k be a random variable equal to the number of isolated components of size k in $G_{n,p}$ for $1 \leq k \leq n/2$. By a union bound, we get:

$$\mathsf{P}(\exists \text{isolated component of size } k, 1 \leq k \leq n/2) \leq \sum_{k=1}^{n/2} \mathsf{P}(X_k = 0) \tag{6}$$

Let us evaluate $\mathsf{E}(X_k)$, the expected number of isolated components of size k. We use a union bound to argue that $\mathsf{E}(X_k)$ is at most $\binom{n}{k}$ times the probability that any given set of k vertices forms an isolated component.

We look at the cases where $k \geq 2$. In order for a given set of k vertices to form an isolated component, those k vertices must form a connected component and the vertices must have no edges to the rest of the graph. For the vertices to form a connected component, there must be a tree with $k-1$ edges connecting the vertices. Any given tree will appear with probability p^{k-1}, and the total number of spanning trees on k vertices is k^{k-2}. The probability that no edges go from the k vertices to the rest of the graph is just $(1-p)^{k(n-k)}$. Then in total, we have

$$\mathsf{E}(X_k) \leq \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \tag{7}$$

$$\leq \left(\frac{ne}{k}\right)^k k^{k-2} \left(\frac{\ln n + c}{n}\right)^{k-1} e^{-\left(\frac{\ln n + c}{n}\right)k(n-k)} \tag{8}$$

$$= \frac{ne^k}{k^2} (\ln n + c)^{k-1} e^{-(\ln n + c)k} \tag{9}$$

For $2 \leq k \leq 5$,

$$(9) \leq c_0 n (\ln n + c)^4 e^{-2\ln n - 2c} \tag{10}$$

$$= \frac{(\ln n + c)^4}{n} e^k < \frac{1}{n^{1-o(1)}} \tag{11}$$

For $k > 5$,

$$(9) \leq \tilde{c} n \frac{(\ln n + c)^{k-1}}{n^k} e^k < \frac{1}{n^{1-o(1)}} \tag{12}$$

Thus, we have:

$$\mathsf{P}(\exists \text{isolated component of size } k, 1 \leq k \leq n/2) \leq \mathsf{P}(X_1 = 0) + \sum_{k=2}^{n/2} \mathsf{P}(X_k = 0) \tag{13}$$

$$\leq \mathsf{P}(\exists \text{isolated vertex}) + \frac{1}{n^{1-o(1)}} \tag{14}$$

$$= \mathsf{P}(\exists \text{isolated vertex}) + o(1) \tag{15}$$

Next, we want to argue that the probability that there are isolated vertices in G goes to 0 as n increases. Let X be a random variable equal to the number of isolated vertices in G. Then,

$$\mathsf{P}(X > 0) \leq \mathsf{E}(X) = n(1-p)^{n-1} = ne^{(n-1)\ln(1-p)} = ne^{(n-1)(-\sum_{j=1}^{\infty} \frac{p^j}{j!})} \tag{16}$$

where the first equality is due to linearity of expectation and the fact that a given vertex is isolated with probability $(1-p)^{n-1}$.

$$= ne^{-(n-1)p} + O(np^2) \approx ne^{-\ln n + c}(1+o(1)) \approx e^{-c} \tag{17}$$

As $c \to \infty$, $\mathsf{P}(X > 0) \to 0$, so the probability that G is connected goes to 1. The case for $c \to -\infty$ can be achieved using similar arguments.

Next, we sketch the proof for when c is fixed. First, we recall the definition of a Poisson distribution. A Poisson random variable Z takes integer values, and we have $\mathsf{P}(Z = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for some $\lambda > 0$ and $\mathsf{E}(Z) = \lambda$. We will prove the following lemma:

Lemma 1.3. *X has an asymptotically Poisson distribution with parameter $\lambda = e^{-c}$.*

Then we have $\lim_{n\to\infty} \mathsf{P}(X = 0) = e^{-\lambda} = e^{-e^{-c}}$. □

Proof.(Proof of Lemma 1.3) We use the proof from [1]. We want to show that $\lim_{n\to\infty} \mathsf{P}(X = k) = \frac{e^{-ck}}{k!} e^{-e^{-c}}$ for all k. Write X as the sum of indicator variables I_v, where for each vertex v, I_v is 1 if v is isolated in $G_{n,p}$ and 0 otherwise. Then we can get an expression for $\mathsf{E}(X)$,

$$\mathsf{E}(X) = \sum_v I_v = n(1-p)^{n-1} = ne^{(n-1)\ln(1-p)} = ne^{-(n-1)\sum_{k=1}^{\infty} \frac{p^k}{k}} \tag{18}$$

$$= ne^{-(n-1)p + O(np^2)} = ne^{-(\log n + c) + O\left(\frac{(\log n)^2}{n}\right)} \approx e^{-c} \tag{19}$$

where we used $p = \frac{\log n + c}{n}$ in the last equality.

We use without proof a theorem from [1]:

Theorem 1.4. *Let $S_n = \sum_{i \geq 1} I_i$ be a sequence of random variables, $n \geq 1$, and let $B_k^{(n)} = \mathsf{E}(\binom{S_n}{k})$. Suppose that there exists $\lambda \geq 0$ such that for every fixed $k \geq 1$,*

$$\lim_{n\to\infty} B_k^{(n)} = \frac{\lambda^k}{k!} \tag{20}$$

Then for every $j \geq 0$,

$$\lim_{n\to\infty} \mathsf{P}(S_n = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \tag{21}$$

i.e., S_n converges in distribution to the Poisson distributed random variable with expectation λ.

We claim that $\lim_{n\to\infty} \mathsf{E}\left(\binom{X}{k}\right) = \frac{e^{-ck}}{k!}$. This is left as an exercise to the reader. This claim allows us to use Theorem 1.4 to conclude that X is asymptotically Poisson. □

2 Largest clique in $G_{n,p}$

Theorem 2.1. *Let p be a fixed constant. Then,*

(a) If $k = \lceil 2\log_{1/p} n \rceil$, then whp, $G_{n,p}$ has no clique of size k.

(b) If $k \geq \lceil 2\log_{1/p} n - 3\log_{1/p}\log_{1/p} n \rceil$, then whp, $G_{n,p}$ has a clique of size k.

Proof of (a). Let X_k be the number of cliques of size k in $G_{n,p}$. Then $\mathsf{E}(X_k) = \binom{n}{k} p^{\binom{k}{2}} \leq \left(\frac{ne}{k} p^{\frac{k-1}{2}}\right)^k \leq \left(\frac{e}{k\sqrt{p}}\right)^k$, where we used the fact that $p^{k/2} \leq 1/n$. Thus, $\mathsf{E}(X_k) \to 0$ as $n \to \infty$ because p is constant. $\square$

Proof of (b). Let $k_1 = k - t$.

$$\mathsf{E}(X_{k_1}) \approx \left(\frac{ne}{k_1} p^{\frac{k_1 - 1}{2}}\right)^{k_1} = \left(\frac{ne}{k-t} p^{\frac{k-t-1}{2}}\right)^{k_1} \tag{22}$$

$$\approx \frac{e}{(k-t)p^{\frac{t+1}{2}}} \tag{23}$$

If we let $t > 2\log_{1/p} k$, this expression is greater than 1, which means that in expectation there will be a clique of size k_1. Now we need to show that this occurs with high probability for $k \geq \lceil 2\log_{1/p} n - 3\log_{1/p}\log_{1/p} n \rceil$. We first get a bound on the variance. Let $X_k = \sum_{S \subseteq V : |S| = k} X_S$, where X_S is an indicator random variable that is 1 when S is a clique. Then,

$$\mathsf{Var}(X_k) = \mathsf{E}(X_k^2) - \mathsf{E}(X_k)^2 = \mathsf{E}\left(\left(\sum X_s\right)^2\right) - \mathsf{E}(X_k)^2 \tag{24}$$

$$= \sum_{S,T} \mathsf{E}(X_S X_T) - E(X_S)E(X_T) \tag{25}$$

$$\leq \sum_{S,T : |S \cap T| \geq 2} \mathsf{E}(X_S X_T) \tag{26}$$

where in the last line, we used the fact that for S and T such that $|S \cap T| \leq 1$, X_S and X_T are independent. Now we write $\sum_{S,T:|S\cap T|\geq 2} \mathsf{E}(X_S X_T)$ as a sum over the size of $S \cap T$. We choose k vertices to form S, then we choose j of those k vertices to form the intersection. Finally, we choose $k - j$ vertices from the remaining $n - k$ vertices to form the rest of T. The probability that k-sets S and T with intersection of size j form cliques is $p^{\binom{k}{2}} p^{\binom{k}{2} - \binom{j}{2}}$. We use this to evaluate the following:

$$\frac{\sum_{S,T:|S\cap T|\geq 2} \mathsf{E}(X_S X_T)}{\mathsf{E}(X_k)^2} = \frac{\sum_{j=2}^{k} \binom{n}{k}\binom{k}{j}\binom{n-k}{k-j} p^{\binom{k}{2}} p^{\binom{k}{2} - \binom{j}{2}}}{\mathsf{E}(X_k)^2} \tag{27}$$

$$= \sum_{j=2}^{k} \frac{\binom{n}{k}\binom{k}{j}\binom{n-k}{k-j} p^{\binom{k}{2}} p^{\binom{k}{2} - \binom{j}{2}}}{\left(\binom{n}{k} p^{\binom{k}{2}}\right)^2} \tag{28}$$

$$= \sum_{j=2}^{k} \frac{\binom{k}{j}\binom{n-k}{k-j}}{\binom{n}{k} p^{\binom{j}{2}}} = \sum_{j=2}^{k} a_j \tag{29}$$

Let us evaluate the a_j terms.

$$a_2 = \frac{k(k-1)}{2} \frac{\binom{n-k}{k-2}}{\binom{n}{k} p} \leq \frac{k^2 \cdot k(k-1)}{n^2 p} \ll 1 \tag{30}$$

$$\frac{a_{j+1}}{a_j} = \frac{\binom{k}{j+1}\binom{n-k}{k-j-1}}{\binom{k}{j}\binom{n-k}{k-j} p^j} = \frac{(k-j)(k-j)}{(j+1)(n-2k+j+1)p^j} \leq \frac{k^2}{(j+1)(n-k)p^j} \tag{31}$$

We know that $j \le k = \lceil 2\log_{1/p} n - 2\log_{1/p} k \rceil$. Then $a_{j+1}/a_j < 1$, so all a_j can be bounded by a_2. Then $\sum_j a_j \le k a_2 \le \frac{k^5}{n^2 p} = o(1)$. Thus, we have shown $\frac{\mathsf{Var}(X_k)}{\mathsf{E}(X_k)^2} \le \frac{k^5}{n^2 p}$. We will use the 1-sided Chebyshev bound:

$$\mathsf{P}(X_k \le \mathsf{E}(X_k) - t\sqrt{\mathsf{Var}(X_k)}) \le \frac{1}{1+t^2} \tag{32}$$

Letting $t = \frac{\mathsf{E}(X_k)}{\sqrt{\mathsf{Var}(X_k)}}$ gives:

$$\begin{aligned} \mathsf{P}(X_k \le \mathsf{E}(X_k) - \frac{\mathsf{E}(X_k)}{\sqrt{\mathsf{Var}(X_k)}}\sqrt{\mathsf{Var}(X_k)}) &= \mathsf{P}(X_k \le 0) && (33)\\ &\le \frac{1}{1+\frac{\mathsf{E}(X_k)^2}{\mathsf{Var}(X_k)}} && (34)\\ &= \frac{\mathsf{Var}(X_k)}{\mathsf{E}(X_k^2)} && (35) \end{aligned}$$

Then $\mathsf{P}(X_k = 0) \le \frac{k^5}{n^2 p} \to 0$. So with high probability, the desired result holds. □

We can actually get an even stronger bound on $\mathsf{P}(X_k = 0)$ using the following theorem:

Theorem 2.2. *(Janson) Let $\mathbb{X}$ be a discrete, finite set. Let X be a random subset of $\mathbb{X}$, where each $a \in \mathbb{X}$ is included in X independently with probability p_a. Let $S_1, ..., S_M$ be subsets of $\mathbb{X}$, and let $Y = |\{i : S_i \subseteq X\}|$. Let $\Delta = \sum_{i,j:S_j \cap S_j \neq \emptyset} \mathsf{P}(S_i, S_j \subseteq X)$. Then,*

$$\mathsf{P}(Y \le \mathsf{E}(Y) - t) \le e^{-t^2/\Delta} \tag{36}$$

Namely, we can prove this corollary:

Corollary 2.3. *For $k \ge \lceil 2\log_{1/p} n - 3\log_{1/p}\log_{1/p} n \rceil$, we have $\mathsf{P}(X_k = 0) \le e^{-\frac{n^2 p}{k^5}}$*

Proof. We will use Janson's inequality where $\mathbb{X}$ is the set of all edges in a graph on n vertices, and the random variable X is the edge set of $G_{n,p}$. $S_1, ..., S_M$ are all of the subsets of V of size k. Then Y is the number of cliques of size k. Then $\Delta = \sum_{\substack{S,T:|S\cap T|\ge 2\\ |S|=|T|=k}} \mathsf{E}(X_S X_T)$. Setting $t = \mathsf{E}(Y)$ and using the bound derived earlier gives $\mathsf{P}(Y \le 0) \le e^{-\frac{n^2 p}{k^5}}$. □

3 Chromatic number of $G_{n,p}$

Let $\chi(G_{n,p})$ be a random variable equal to the chromatic number of a graph chosen from $G_{n,p}$. We can get the following rough bound on the chromatic number of $G_{n,p}$.

Theorem 3.1. *With high probability,*

$$\begin{aligned} \frac{n}{\lceil 2\log_{\frac{1}{1-p}} n \rceil} \le \chi(G_{n,p}) &\le \frac{n}{\lceil 2\log_{\frac{1}{1-p}} n - 3\log\log n \rceil} + \frac{n}{(2\log_{\frac{1}{1-p}})^2} && (37)\\ &= \frac{n}{2\log_{\frac{1}{1-p}} n}(1+o(1)) && (38) \end{aligned}$$

Proof. Let $\alpha(G)$ be the size of the largest independent set in a graph G. Note that the size of the largest clique in G is equal to the size of the largest independent set in the complement of G. Then our bounds on the size of the largest clique in $G_{n,p}$ give us bounds on $\alpha(G_{n,1-p})$. Thus, we get the following lemma:

Lemma 3.2. *For a random graph $G_{n,p}$, we have*

$$\lceil 2\log_{\frac{1}{1-p}} n - 3\log_{\frac{1}{1-p}} \log_{\frac{1}{1-p}} n\rceil \leq \alpha(G_{n,p}) \leq \lceil 2\log_{\frac{1}{1-p}} n\rceil \tag{39}$$

with high probability.

From the upper bound in the lemma, we can get the lower bound on $\chi(G_{n,p})$. Each color in a coloring forms an independent set. The lemma shows that no more than $\lceil 2\log_{\frac{1}{1-p}} n\rceil$ vertices can have the same color. Then in total, we have $\chi(G_{n,p}) \geq \frac{n}{\lceil 2\log_{\frac{1}{1-p}} n\rceil}$.

Next, we get the upper bound on $\chi(G_{n,p})$. Note that the Lemma 3.2 holds for random graphs of any size. Let $f(n) = \lceil 2\log_{\frac{1}{1-p}} n - 3\log_{\frac{1}{1-p}} \log_{\frac{1}{1-p}} n\rceil$. Let ν be some parameter we will fix later. We will show the following lemma.

Lemma 3.3. *For $\nu \geq \log^c n$ for some suitable constant c, every subset of $G_{n,p}$ of size at least ν will contain an independent set of size $f(\nu)$ whp as $n \to \infty$.*

Proof. From Corollary 2.3, we have that $\mathsf{P}(X_k = 0) \leq e^{-n^2 p/k^5}$ for suitable k. Let Y_k be the number of independent sets of size k in $G_{n,p}$. Since cliques are independent sets in a graph's complement we immediately get $\mathsf{P}(Y_k = 0) \leq e^{-n^2(1-p)/k^5}$ (since $G_{n,1-p}$ is distributed like the complement of $G_{n,p}$). For each $S \subseteq V$ of size ν, let E_S be the event that S does not contain an independent set of size $f(\nu)$. Then,

$$\mathsf{P}\left(\bigcup_{S\subseteq V, |S|=\nu} E_S\right) \leq \binom{n}{\nu}\mathsf{P}(Y_{f(\nu)} = 0) \tag{40}$$

$$\leq \binom{n}{\nu} e^{-\nu^2(1-p)/f(\nu)^5} << 1 \qquad \text{(for } \nu \geq \log^c n\text{)} \tag{41}$$

Then whp every subset S of size ν will have an independent set of size $f(\nu)$. ☐

Given an instance of $G_{n,p}$ we pick a subset of ν vertices and remove an independent set S of size $f(\nu)$ from it. We pick another subset from $V \setminus S$ and remove an independent set of size $f(\nu)$ from it. We can keep picking subsets of size ν from the remaining vertices and removing independent sets of size $f(\nu)$ from each of those subsets until there are fewer than ν vertices left. Note that Lemma 3.3 implies that whp all of the subsets of size ν that we picked will contain independent sets of size $f(\nu)$. Call the set of remaining vertices T. We assign one color to each independent set and one color for each vertex in T. This gives an upper bound on $\chi(G_{n,p})$.

In particular, we could have removed as many as $\frac{n}{f(\nu)}$ independent sets, so $\chi(G_{n,p}) \leq \frac{n}{f(\nu)} + \nu$ with high probability (the high probability ensures that each subset had a suitably-sized independent set). If we set $\nu = \frac{n}{\log_{\frac{1}{1-p}} n}$, we get $\chi(G_{n,p}) \leq \frac{n}{2\log_{\frac{1}{1-p}} n}(1 + o(1))$. ☐

4 Sharper bounds on $\chi(G_{n,p})$

We use the following concentration inequality:

Theorem 4.1. *(McDiarmid's inequality) Let $x_1, .., x_n$ be independent random variables such that each x_i belongs to the set A_i. Let $f : A_1 \times A_2 \times \cdots \times A_n \to \mathbb{R}$ be some function. Let $\vec{x} = (x_1, ..., x_n)$ signify a vector of x_i's. Assume that if $\vec{x}$ and $\vec{x}'$ differ only in the i^{th} coordinate, then $|f(\vec{x}) - f(\vec{x}')| \leq c_i$. Then,*

$$\mathsf{P}(|f(x_1, ..., x_n) - \mathsf{E}(f(\vec{x}))| > t) \leq 2e^{-\frac{2t^2}{\sum_i c_i^2}} \tag{42}$$

Now we prove the following claim:

Claim 4.2. $\mathsf{P}(|\chi(G_{n,p} - \mathsf{E}(G_{n,p})| > \sqrt{n}w(1)) \to 0$

where $w(1)$ signifies any function that goes to infinity as n goes to infinity.

We prove this using a corollary of McDiarmid's inequality.

Corollary 4.3. *If $A_1, ..., A_m$ is a partition of the edges of the complete graph K_n and $f(G)$ is such that $|f(G) - f(G')| \leq 1$ if $E(G)\Delta E(G') \subseteq A_i$, where $A\Delta B = (A \setminus B) \cup (B \setminus A)$, then $\mathsf{P}(|f(G) - \mathsf{E}(f(G))| > t) \leq 2e^{-\frac{2t^2}{m}}$.*

To prove the claim, let $\chi(G) = f(G)$ and number the vertices 1 through n. Then let $A_i = \{(j, i) : j < i\}$ for some arbitrary orientation of the edges. Note that if we change edges in only one A_i, we only affect edges adjacent to vertex i. Removing or adding edges connected to vertex i can only change the chromatic number by at most 1, so we can use the corollary, which immediately gives $\mathsf{P}(|\chi(G_{n,p}) - \mathsf{E}(\chi(G_{n,p}))| > \sqrt{n}w(1)) \leq 2e^{-2w(1)^2} \to 0$.

We will move onto an even stronger statement about the chromatic number of a random graph. First, we have this theorem:

Theorem 4.4. *(Alon-Krivelevich) Let $p = n^{-\alpha}$ for $\alpha \in (1/2, 1)$. Then with high probability $\chi(G_{n,p})$ takes one of two consecutive integer values.*

We will not prove this result, but instead will prove the following weaker result:

Theorem 4.5. *Let $p = n^{-\alpha}$, $\alpha \in (5/6, 1)$. Then with high probability, $\chi(G_{n,p})$ takes one of four consecutive integer values.*

Proof. Let $u(n)$ be the least integer such that $\mathsf{P}(\chi(G_{n,p}) \leq u(n)) \geq \frac{1}{\log n}$. Let Y be the size of the smallest set W such that $G \setminus W$ is $u(n)$ colorable.

We will number the vertices from 1 to n and use the same partition of A_i's as before, namely letting $A_i = \{(j, i) : j < i\}$ for some arbitrary orientation of the edges. Then by Corollary 4.3, we have $\mathsf{P}(|Y - \mathsf{E}(Y)| > \sqrt{n}w(1)) \leq e^{-w(1)^2} \to 0$. When $Y = 0$, by definition this means that G is $u(n)$ colorable. This happens with probability at least $\frac{1}{\log n}$. Thus, $\mathsf{P}(Y = 0) \geq \frac{1}{\log n}$.

We know that Y is exponentially concentrated around its expectation. In particular, we can use Corollary 4.3 to get $\mathsf{P}(|Y - \mathsf{E}(Y)| > \sqrt{n \log n}) < 2n^{-2}$. For sufficiently large n, $2n^{-2} < \frac{1}{\log n}$, so we must have $\mathsf{E}(Y) \leq \sqrt{n \log n}$ or else we wouldn't be able to have $\mathsf{P}(Y = 0) \geq \frac{1}{\log n}$.

Moreover, we've shown that $\mathsf{P}(|Y - \mathsf{E}(Y)| > \sqrt{n \log n}) < 2n^{-2} \to 0$ as $n \to \infty$. Then since $\mathsf{P}(Y > 2\sqrt{n \log n}) \leq \mathsf{P}(|Y - \mathsf{E}(Y)| > \sqrt{n \log n})$, we have $\mathsf{P}(Y > 2\sqrt{n \log n}) \to 0$ as $n \to \infty$.

Then for sufficiently large n, $Y \leq 2\sqrt{n \log n}$ with high probability. By the definition of Y this means that there is some set W of vertices of size $\leq 2\sqrt{n \log n}$ such that $G \setminus W$ is $u(n)$ colorable.

To color G, we will remove such a set W and color $G \setminus W$ with $u(n)$ colors. Then we will use the following lemma:

Lemma 4.6. *Let α and c be fixed constants, with $\alpha \in [\frac{5}{6}, 1], p = n^{-\alpha}$, then asymptotically almost surely, every $S \subseteq V$ of size $\leq C\sqrt{n \log n}$ is 3-colorable.*

Lemma 4.6 shows that W is 3-colorable with high probability for large n, so we will need at most 3 additional colors to color W. Then in total, $\chi(G_{n,p}) \in \{u(n), u(n)+1, u(n)+2, u(n)+3\}$. □

Proof of Lemma4.6. Let T be a minimal subset of V that is not 3-colorable. For the sake of contradiction, assume $|T| \leq C\sqrt{n \log n}$. By minimality of T, removing any vertex from T will cause it to be 3-colorable. Thus, all vertices in T must have degree at least 3 in the subgraph induced by T. If a vertex v had degree 2 or less, we could get a 3-coloring for $T \setminus \{v\}$ and then use that 3-coloring for T by coloring v with a color not used in by one of v's neighbors in T.

Therefore, the number of edges in T is at least $\frac{3|T|}{2}$. Then let E_t be the event that there exists a non-3-colorable $T \subseteq V$ of size t. The probability that any given set of t vertices forms a non-3-colorable graph is at most the probability that there exists a subset of t vertices with at least $\frac{3t}{2}$ edges.

$$\mathsf{P}(\cup_{t=1,\ldots,C\sqrt{n \log n}} E_t) \leq \sum_{t=1}^{C\sqrt{n \log n}} \binom{n}{t} \binom{\binom{t}{2}}{\frac{3t}{2}} p^{\frac{3t}{2}} \to 0 \tag{43}$$

where we have used a union bound over all possible sizes of T. For each value of t, there are $\binom{n}{t}$ possible subsets of t vertices and each subset has $\binom{\binom{t}{2}}{\frac{3t}{2}}$ possible ways to choose $\frac{3t}{2}$ edges. Thus, we have shown that as $n \to \infty$, the number of subsets of V of size at most $2\sqrt{n \log n}$ that are not 3-colorable goes to 0. □

References

[1] Alan Frieze and Michał Karoński. *Introduction to Random Graphs.* Cambridge University Press, 2015.