High Dimensional Geometry

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"The Geometric Perspective is often an insightful one."
The main question we will address in this module is:
"What is the structure of high-dimensional sets or distributions that makes it possible to have efficient algorithms?"

For example, in optimization, convexity is the key property. We seek to identify a similar fundamental property in high dimensional sets. We start by computing the volume for standard geometric bodies. For a cubic shape $V_{cube} = a^n$, and for a sphere $V_{ball} = \frac{4}{3}r^3$. In n dimensional space $V_{ball} = C_n r^n$) where $C_n \approx \frac{c}{(\sqrt{n})^n} r^n$, so $V_{Bn} \approx (\frac{c}{\sqrt{n}})^n$. Therefore the ratio of the two volumes is $(\frac{1}{\sqrt{n}})^n$, and to cover a cube with balls it would take $(n)^{n/2}$ such balls. Now if the ball radius is reduced slightly from 1 to $1 - \epsilon$ as shown in Figure 1, the fraction of the original volume contained in the smaller ball is

$$Vol(B((1-\epsilon)r))/Vol(B(r) = (1-\epsilon)^n$$

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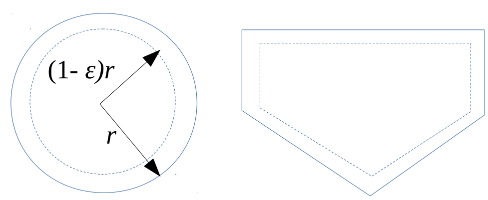

FIGURE 1: Cap formed by slice of sphere

For a shell of constant thickness t, what is t that contains half the unit ball volume? $(1-\epsilon)^n < \frac{1}{2}$, or $\epsilon \approx \frac{1}{n}$. So we see that the volume in higher dimensional space is concentrated in a thin layer at the surface of the ball. Note that this also applies to any other set with volume, such as the polygon shown in Figure 1. So in summary, the set volume is very close to the boundary.

Now, instead of examining shell volume, we examine cap volume for a ball of radius one. What fraction of the ball volume is in a cap with offset t, as shown in figure 2? Note that upon slicing B_n, the cross section is a ball B_{n-1} of dimension n-1 with radius $\sqrt{1-t^2}$. Now defining A_0 as the resulting ball volume for t=0, and A_t as the volume of ball resulting from the $t > 0$ cut, we have

$$\frac{Vol_{n-1}(A_t)}{Vol_{n-1}(A_0)} = (\sqrt{1-t^2})^{n-1} \leq e^{-t^2(n-1)/2}$$

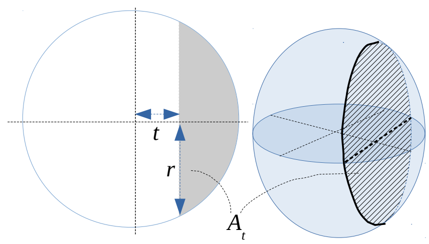

FIGURE 2: Cap formed by a slice through a sphere

where the last inequality is the result of applying $(1+x) < e^x$. Integrating to find the entire cap volume relative the entire half-volume of $B(1)$ yields

$$\frac{Vol(Cap(t))}{\frac{1}{2}Vol(B(1))} = \frac{\int_t^1 Vol(A_t)dt}{\int_0^1 Vol(A_t)dt}$$

At which t does the cap relative volume become $\approx \frac{1}{2}$? This occurs for $t \approx \frac{1}{\sqrt{n}}$. For $t = 2/\sqrt{n}$ the relative volume is $1/e$. Notice that $\forall t > \frac{1}{\sqrt{n}}$, the ratio drops as $e^{-t^2(n-1)/2}$, which is similar to bounds on the tail of a gaussian density from t to ∞.

So in summary, within a ball B_1 of radius 1, a constant fraction of the volume lies within a $\sqrt{n}$ band as is true for other bands. As shown in Figure 3, for the slice of thickness $2\ell/\sqrt{n}$, the volume V_o outside the band is bounded as $V_o \le e^{-\ell^2/2}$. This leads to the question: Where is the volume? Is it in the boundary or in the middle (i.e. *any* middle) of the ball? In $\mathbb{R}^2$ the intersection of two bands seems to contain most of the volume in the middle. However for large n, most of the intersection volume is contained in the boundary. For the ball shown in Figure 3, most of the intersection volume exists in the boundary patches indicated by the shaded regions.

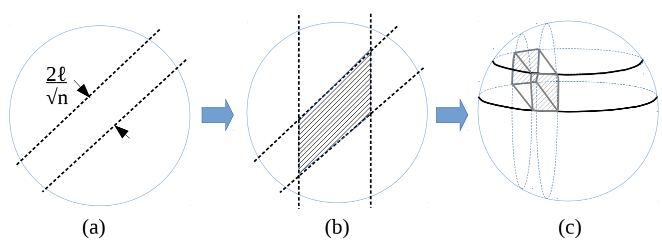

FIGURE 3: (a)Band construction; (b)For small n, most of intersection volume lines in middle; (c) For large n, most of intersection volume lies in two patches on the boundary

For a final example, consider a slice through a cube as shown in Figure 4. We would like to know the mass distribution in an arbitrary direction a. Once again, by taking a smaller cube of side $(1-\epsilon)$, we see by the earlier argument that most of the mass is on the boundary as before. However, notice that in contrast to the previous example of slicing a ball, the slice is *not* a cube (aside: by taking a slice of a cube, we can generate any symmetric convex polytope). Nevertheless, we can still ask the question: How does the volume outside the band decay with distance along a? (i.e. does the cube also have most of the mass in the middle?) Formally, for picking a random x from the cube, we seek to know what is $\mathcal{P}(a \cdot x)$.

Note that the coordiates x_i of x are independent, the terms $a_i x_i$ are uniform and bounded as $-a_i < a_i x_i < a_i$, so that regardless of the unit vector a direction,

$$\mathcal{P}(\sum a_i x_i > t) \leq e^{-\frac{2t^2}{\sum a_i x_i}} = e^{-\frac{t^2}{2\sum a_i^2}} \leq e^{-t^2/2}$$

So we see that the volume outside the band decays quickly. Note that this is an upper bond because by a distance $\sqrt{n}$, no volume remains (not e^{-n}, as with the ball). This exponential relationship appears for the gaussian tail, the ball, cube and for *any* convex body.

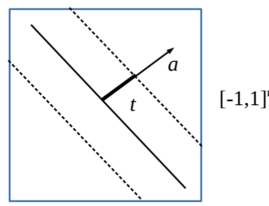

Figure 4: A band through a cube

Next, we look at a general formulation that controls this volume distribution. This formulation will help us determine if there exist efficient algorithms to solve these high dimensional problems. If the volume distribution is irregular with wide variations, it becomes difficult to implement efficient algorithms. We will therefore assume a uniform random distribution of points x in the high dimensional space.

Definition 1. For A, B compact sets in $\mathbb{R}^n$ that have volume, we define the $+$ operation as $A+B = x + y, \quad x \in A, y \in B$, and we define $(1-\lambda)A+\lambda B = (1-\lambda)x + \lambda y, \quad x \in A, y \in B$ as the *Minkowski sum*.

Two examples of the Minkowski sums for arbitrary sets are shown in 5.

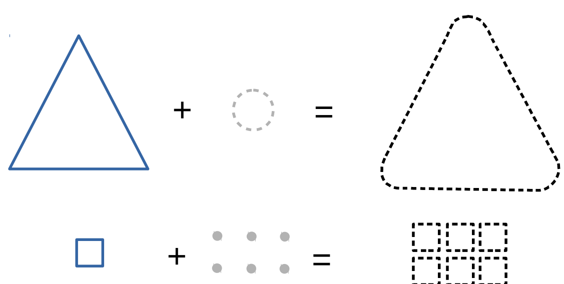

Figure 5: Two examples of Minkowski sums

We can now state the following theorem, which was proved for $n \leq 3$ by Brunn, and for $n > 3$ by Minkowski. We start with the convex combination of two sets

$$[Vol((1-\lambda)A + \lambda B)]^{1/n} \geq (1-\lambda)Vol(A)^{1/n} + \lambda Vol(B)^{1/n}$$

By substituting $A = (1-\lambda)A, B = (\lambda)B$, we have the following theorem.

Theorem 2 (Brunn-Minkowski). *For A,B arbitrary sets in* $\mathbb{R}^n$ *that have volume,* $[Vol(A+B)]^{1/n} \geq Vol(A)^{1/n} + Vol(B)^{1/n}$.

As inspiration for the proof, consider three sections of an arbitrary convex body in $\mathbb{R}^n$ as shown in Figure 6. Let A_z in $\mathbb{R}^{n-1}$ be the section of the convex body at point z. We make the following claim:

Claim 3.

$$[Vol(A_{\frac{x+y}{2}})]^{\frac{1}{n-1}} \geq \frac{1}{2}[Vol(A_x)]^{\frac{1}{n-1}} + \frac{1}{2}[Vol(A_y)]^{\frac{1}{n-1}}$$

$$[Vol(A_{\frac{x}{2}+\frac{y}{2}})]^{\frac{1}{n-1}} \geq \frac{1}{2}[Vol(A_x)]^{\frac{1}{n-1}} + \frac{1}{2}[Vol(A_y)]^{\frac{1}{n-1}}$$

Note that $volume^{\frac{1}{n}}$ is a *concave* function. So we have

$$[Vol(\frac{1}{2}A_x + \frac{1}{2}A_y)]^{\frac{1}{n-1}} \geq \frac{1}{2}[Vol(A_x)]^{\frac{1}{n-1}} + \frac{1}{2}[Vol(A_y)]^{\frac{1}{n-1}}$$

Now, since the points in A_x and A_y are points in a convex body, their convex combination is also in the body, and

$$Vol(A_{\frac{x+y}{2}}) \geq \tfrac{1}{2}Vol(A_x) + \tfrac{1}{2}Vol(A_y)$$

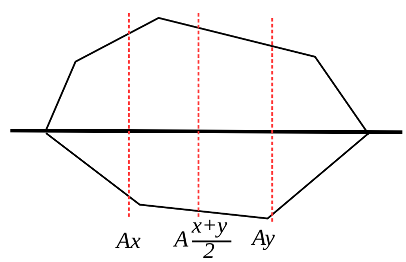

FIGURE 6: Three sections in an arbitrary convex body in $\mathbb{R}^n$

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Now, replace each section with a ball of the same volume to produce a centered body as shown in Figure 7. The radius of the ball at x grows with the volume, so the radius $r(x) \approxeq Vol(x)^{\frac{1}{n-1}}$, and is a *concave* function.

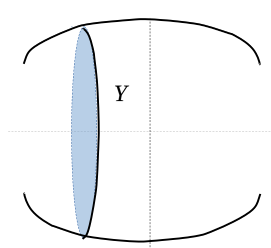

FIGURE 7: Replacing each slice with a ball of same volume creates a symmetric body

To prove these statements, we first consider the easy case of cuboids A, B with side lengths a, b. The Minkowski sum A+B is shown in Figure 8.

So we must check the inequality

$$Vol(A+B) = \Pi(a_i + b_i)^{\frac{1}{n}} \geq (\Pi a_i)^{\frac{1}{n}} + (\Pi b_i)^{\frac{1}{n}}$$

$$1 \geq \frac{(\Pi a_i)^{\frac{1}{n}} + (\Pi b_i)^{\frac{1}{n}}}{\Pi(a_i + b_i)^{\frac{1}{n}}}$$

$$1 \geq \Pi\left(\frac{a_i}{a_i + b_i}\right)^{\frac{1}{n}} + \Pi\left(\frac{b_i}{a_i + b_i}\right)^{\frac{1}{n}}$$

By the geometric mean inequality,

$$\begin{aligned} 1 &= \frac{1}{n}\sum\left(\frac{a_i}{a_i + b_i}\right) + \frac{1}{n}\sum\left(\frac{b_i}{a_i + b_i}\right) \\ &\geq \Pi\left(\frac{a_i}{a_i + b_i}\right)^{\frac{1}{n}} + \Pi\left(\frac{b_i}{a_i + b_i}\right)^{\frac{1}{n}} \\ &\geq \frac{(\Pi a_i)^{\frac{1}{n}} + (\Pi b_i)^{\frac{1}{n}}}{\Pi(a_i + b_i)^{\frac{1}{n}}} \\ &\geq \frac{(\Pi a_i)^{\frac{1}{n}} + (\Pi b_i)^{\frac{1}{n}}}{Vol(A+B)} \end{aligned}$$

$$Vol(A+B) \geq (\Pi a_i)^{\frac{1}{n}} + (\Pi b_i)^{\frac{1}{n}} \tag{1}$$

So we see Theorem 2 is true for cuboids as the convex body.

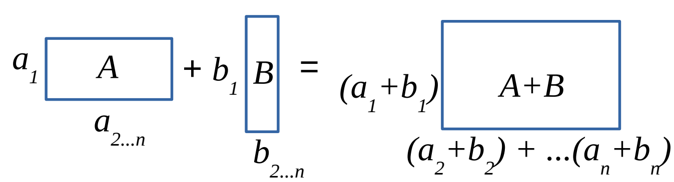

FIGURE 8: Minkowski sum of two cuboids in $\mathbb{R}^n$

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Now we examine an arbitrary body that has volume. For example, consider the volume of A as the disjoint countable union of boxes as shown in Figure 9. We bound the volume above by including those boxes that intersect the boundary and bound it below by only including the boxes in the interior of A. By letting the box edge length goto zero, we can compute the volume of A. So it suffices to consider, instead of general bodies, that A and B are disjoint unions of boxes. Furthermore, we allow A and B to overlap. If we can prove the result for this case, then we take the limit and the equality holds at each step, and thereby prove the theorem for general bodies.

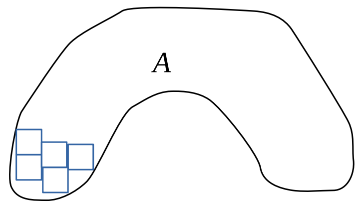

FIGURE 9: Arbitrary body as sum of disjoint cuboids in $\mathbb{R}^n$
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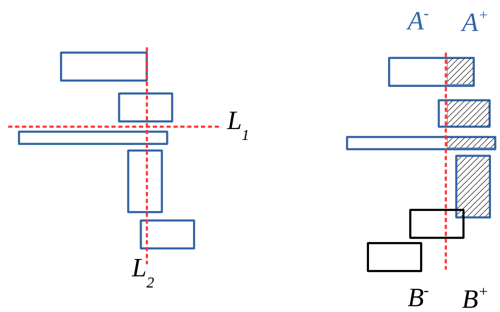

FIGURE 10: Axis aligned boxes and cuts $\mathbb{R}^n$
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So to solve this problem for an arbitrary countable union of boxes, as shown in 10, we will use induction on the total number of boxes for sets A and B. We begin with the following claim.

Claim 4. *Given a arbitrary finite collection of boxes, there exists an axis-parallel cut such that there is a complete box on each side of the cut.*

In Figure 10, the L_2 cut does not satisfy the claim, however, by searching another axis-parallel direction, we see that L_1 does satisfy the claim. Before proving the Claim, we introduce the following Proposition.

Proposition 5. *If in some dimension no such cut exists, then all boxes contain a common point in that dimension.*

Proof. Since no such cut exists, then every pair of boxes contains some common point in that dimension, otherwise those two boxes could be separated in that dimension. Therefore, since every pair of boxes overlap in that dimension, the intersection of their union must be nonempty in that dimension and there must be a point common to all boxes in that dimension. □

We now prove the Claim.
Proof. For those boxes that overlap, partition each such box into two pieces, labeled A^-, A^+ as shown in Figure 10. We ignore the rest of the boxes in A for the moment. The point of this partition is that each side has one less box. We also split any boxes B that overlap the cut into B^-, B^+. We then translate B so that the following inequality holds. Such a

cut can be found because a cut can be placed along the dimension of interest so that both $Vol(A^-), Vol(B^-)$ are zero or both $Vol(A^-), Vol(B^-)$ are one, and the volume of each is a continuous function along that dimension.

$$\frac{Vol(A^-)}{Vol(B^-)} = \frac{Vol(A^+)}{Vol(B^+)}$$

We form the Minkowski sums of the total collection,

$$Vol(A + B) \geq Vol(A^- + B^-) + Vol(A^+ + B^+)$$

and by the induction hypothesis,

$$\geq [Vol(A^-)^{\frac{1}{n}} + Vol(B^-)^{\frac{1}{n}}]^n + [Vol(A^+)^{\frac{1}{n}} + Vol(B^+)^{\frac{1}{n}}]^n$$

$$\geq Vol(B^-) \left(\frac{Vol(A^-)^{\frac{1}{n}}}{Vol(B^-)^{\frac{1}{n}}} + 1 \right)^n + Vol(B^+) \left(\frac{Vol(A^+)^{\frac{1}{n}}}{Vol(B^+)^{\frac{1}{n}}} + 1 \right)^n$$

$$\geq Vol(B) \left(\frac{Vol(A^-)^{\frac{1}{n}}}{Vol(B^-)^{\frac{1}{n}}} + 1 \right)^n$$

However, since the ratios $Vol(A^-)/Vol(B^-) = Vol(A^+)/Vol(B^+)$ and A, B are disjoint, then the ratio is preserved over the entire volumes, so

$$\geq \left(Vol(B)^{\frac{1}{n}} \right)^n \left(\frac{Vol(A)^{\frac{1}{n}}}{Vol(B)^{\frac{1}{n}}} + 1 \right)^n$$

$$Vol(A + B) \geq \left(Vol(A)^{\frac{1}{n}} + Vol(B)^{\frac{1}{n}} \right)^n$$

□

We now generalize this result to an equivalent theorem for functions.

Theorem 6 (Prékopa-Leindler). *Let $\lambda \in [0, 1]$, suppose the functions f,g,h : $\mathbb{R}^n \mapsto [0, \infty)$ satisfy*

$$h((1 - \lambda)x) + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$$

for all $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h \geq (\int_{\mathbb{R}^n} f)^{1-\lambda} (\int_{\mathbb{R}^n} g)^\lambda.$$

In the special case of $f = \mathbb{1}_A, g = \mathbb{1}_B, h = \mathbb{1}_{(1-\lambda)A+\lambda B}$, the integrals in $\mathbb{R}^n$ yield volumes and the inequality is equivalent to Brunn-Minkowski, i.e.,

$$Vol((1 - \lambda)A + \lambda B) \geq Vol(A)^{1-\lambda}\, Vol(B).^\lambda$$

For the proof, we first define the level set $L_h(t) = \{x : h(x) \geq t\}$,
Proof. Let n = 1, and suppose f, g, h are bounded, and by scaling f and g, sup f = sup g = 1. For h, consider the level set $L_h(t) = \{x : h(x) \geq t\}$, then

$$\int_{\mathbb{R}^n} h = \int_0^1 Vol(L_h(t))dt$$

Now consider the level sets for f, g at the same value t, then $f(x) \geq t$, $g(x) \geq t$, so

$$t^{1-\lambda} t^{\lambda} = h(x) \geq t$$

.

and we form the integral of h using level sets as shown in Figure 11.

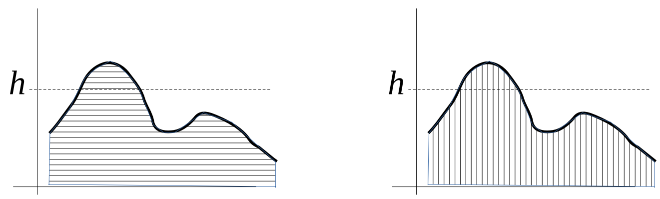

FIGURE 11: Integration can also be performed via level sets

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Therefore, $L_h(t)$ contains the convex combinations of f, g, i.e.

$$L_h(t) \geq (1-\lambda)L_f(t) + \lambda L_g(t)$$

$$\int_{\mathbb{R}} h \geq \int_0^1 Vol((1-\lambda)L_f(t) + \lambda L_g(t))dt$$

and by Brunn-Minkowski,

$$= (1-\lambda)\int_0^1 Vol(L_f(t)) + \lambda \int_0^1 Vol(L_g(t))dt$$

and because f, g are bounded by 1,

$$= (1-\lambda)\int f + \lambda \int g$$

but the arithmetic average is bounded below by the geometric average, so

$$\geq \quad \left(\int f\right)^{(1-\lambda)} + \left(\int g\right)^{\lambda}.$$

So we have used Brunn-Minkowski for the 1-D case. Now, we consider the general case for $n > 1$, and we wish to use induction on the dimension.

Define $x, y \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{R}$, $\gamma = (1-\lambda)\alpha + \lambda\beta$. and $h(x, \alpha) = h_\alpha(x)$, for fixed α. Note that h is defined on $\mathbb{R}^n$), and we are fixing one of the coordinates, so we define a new function $h_\gamma(x) \doteq h(x, \gamma)$, where $h_\gamma : \mathbb{R}^{n-1} \mapsto \mathbb{R}_+$. Note that we are trying to confirm the condition holds for the $(n-1)$ dimensional function as part of the induction proof. So for fixed β, and therefore fixed γ, we have

$$\begin{aligned}
h_\gamma((1-\lambda)x + \lambda y) &= h((1-\lambda)x + \lambda y, (1-\lambda)\alpha + \lambda\beta) \\
&= h((1-\lambda)(x, \alpha) + \lambda(y, \beta)) \\
&\geq f(x, \alpha)^{1-\lambda} g(y, \beta)^{\lambda} \\
&= f_\alpha(x)^{1-\lambda} g_\beta(y)^{\lambda}
\end{aligned} \tag{2}$$

where the inequality is due to the induction hypothesis. Therefore, by the induction hypothesis applied to $h_\gamma, f_\alpha, g_\beta$,with fixed γ, which satisfy the assumptions of the hypothesis for dimension $n-1$,

$$\underbrace{\int_{\mathbb{R}^{n-1}} h_\gamma}_{H(\gamma)} \geq \underbrace{\int_{\mathbb{R}^{n-1}} (f_\alpha)^{1-\lambda}}_{F(\alpha)^{1-\lambda}} \quad \underbrace{\int_{\mathbb{R}^{n-1}} (g_\beta)^{\lambda}}_{G(\beta)^{\lambda}} \tag{3}$$

So now, how do we use this result to prove the theorem for n dimensions?
Since $\gamma = \alpha + \beta$,

$$H((1-\lambda)\alpha + \lambda\beta) \geq F(\alpha)^{1-\lambda} G(\beta)^{\lambda}$$

and by using the 1-D case of the Theorem,

$$\int_{\mathbb{R}} H \geq \left(\int_{\mathbb{R}} F\right)^{1-\lambda} \left(\int_{\mathbb{R}} G\right)^{\lambda}$$

but h is itself an integral, so

$$\int_{\mathbb{R}^n} h \geq \left(\int f\right)^{1-\lambda} \left(\int g\right)^{\lambda}$$

□

Since Theorem 6 is stated for functions, it has more applications than Theorem 2, as we will see. But first, we will introduce the notion of logconcave functions.

Definition 7. We say that a function $f : \mathbb{R}^n \to [0, \infty)$ is *logconcave* if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$.

Remark 8. If range of f is $(0, \infty)$, then f is logconcave iff logarithm of f is a concave function.

Example 9 (Examples of logconcave functions).

1. An indicator function $\mathbb{1}_S : \mathbb{R}^n \to \{0,1\}$ of a subset $S \subseteq \mathbb{R}^n$ is logconcave iff S is connected and convex.

2. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is any convex function, then $f(x) := e^{-\phi(x)}$ is a logconcave function. In particular, the Gaussians $\gamma_n = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{\|x\|^2}{2}}$ are logconcave.

3. If f, g are logconcave functions, then so are $\min\{f, g\}$,fg (however $f + g$ need not).

4. If $f : \mathbb{R}^n \to \mathbb{R}$ is a logconcave function, and for k with $1 \leq k \leq n$ let us define the projection $F : \mathbb{R}^k \to \mathbb{R}$ by

$$(a_1, ..., a_k) \mapsto \int_{x_{k+1},...,x_n} f(a_1, ..., a_k, x_{k+1}, ..., x_n)$$

then by applying Theorem 6, we see that F is also logconcave. Thus projections of logconcave functions are logconcave.

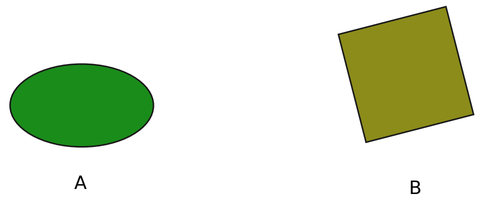

FIGURE 12: $\mathbb{1}_A$ and $\mathbb{1}_B$ are logconcave, but $\mathbb{1}_A + \mathbb{1}_B$ is not.

Definition 10. We say that a function $f : S^{n-1} \to \mathbb{R}$ is *L-Lipshitz* if

$$|f(x) - f(y)| \leq L d_{S^{n-1}}(x, y)$$

where $d_{S^{n-1}}$ denotes arclength on the sphere S^{n-1}.

Theorem 11 (Levy). *If $f : S^{n-1} \to \mathbb{R}$ is L-Lipshitz, then*

$$\mathbb{P}_{S^{n-1}}(|f - \mathbb{E}f| > t\mathbb{E}f) \leq e^{-\frac{t^n}{2L^2}}$$

This means that functions that are L-Lipschitz on the sphere are heavily concentrated near their expectation.

An application of Theorem 6 is the following concentration inequality, which states that if a set has more than half the volume, if you slightly enlarge the set, it will have almost all the volume. Let us denote the complement of a subset S in $\mathbb{R}^n$ by $\overline{S}$.

Theorem 12 (Levy's inequality). *Let $A \subset \mathbb{R}^n$, and let $\gamma = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{\|x\|^2}{2}}$ be the standard Gaussian distribution on $\mathbb{R}^n$. Let us define $A_t := \{x \in \mathbb{R}^n : d(x, A) \leq t\}$, where $d(x, A) := \inf_{a \in A} \|x - a\|_2$. If $\gamma(A) \geq \frac{1}{2}$, then for any $t \in [0, \infty)$ we have $\gamma(\overline{A_t}) \leq 2e^{-\frac{t^2}{4}}$.*

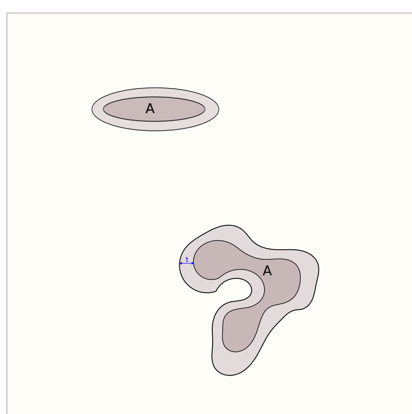

FIGURE 13: A region of $\mathbb{R}^n$ containing A and A_t

Proof. We will need the following:

Claim 13.

$$\gamma(A)\int_{\mathbb{R}^n} e^{\frac{d(x,A)^2}{4}}d\gamma(x)\leq 1$$

We will use Theorem 6 with $\lambda=\frac{1}{2}$ to prove the claim. Let us define functions $f,g,h:\mathbb{R}^n\to[0,\infty)$ by the formulas $f(x)=\mathbb{1}_A(x)\gamma(x)$, $g(x)=e^{\frac{d(x,A)^2}{4}}\gamma(x)$, $h(x)=\gamma(x)$.

Let us check the hypothesis of Theorem 6 when $\lambda=\frac{1}{2}$. Note that if $x\notin A$ then $f(x)=0$ and the inequality is vacuous. So we may assume that $x\in A$, and in this case we have $d(y,A)\leq\|x-y\|$. Hence for all $x\in A$, $y\in\mathbb{R}^n$ we have:

$$\begin{aligned}
&f(x)^{\frac{1}{2}}g(y)^{\frac{1}{2}}\\
&=(\mathbb{1}_A(x)\gamma(x))^{\frac{1}{2}}(e^{\frac{d(y,A)^2}{4}}\gamma(y))^{\frac{1}{2}}\\
&=\frac{1}{(\sqrt{2\pi})^n}(e^{-\frac{\|x\|^2}{2}}e^{\frac{d(x,A)^2}{4}}e^{-\frac{\|y\|^2}{2}})^{\frac{1}{2}}\\
&\leq\frac{1}{(\sqrt{2\pi})^n}(e^{\frac{\|x-y\|^2}{4}-\frac{\|x\|^2}{2}-\frac{\|y\|^2}{2}})^{\frac{1}{2}}\\
&=\frac{1}{(\sqrt{2\pi})^n}(e^{-\frac{\|x+y\|^2}{4}})^{\frac{1}{2}}\\
&=\frac{1}{(\sqrt{2\pi})^n}e^{-\frac{\|\frac{x+y}{2}\|^2}{2}}\\
&=h(\frac{x+y}{2})
\end{aligned}$$

The claim follows since $\int_{\mathbb{R}^n}f=\gamma(A)$, $\int_{\mathbb{R}^n}g=\int_{\mathbb{R}^n}e^{\frac{d(x,A)^2}{4}}d\gamma(x)$, $\int_{\mathbb{R}^n}h=1$.

Let us now use claim to complete the proof:

$$e^{\frac{t^2}{4}}\gamma(\overline{A_t})=\int_{\overline{A_t}}e^{\frac{t^2}{4}}d\gamma(x)\leq\int_{\overline{A_t}}e^{\frac{d(x,A)^2}{4}}d\gamma(x)\leq\int_{\mathbb{R}^n}e^{\frac{d(x,A)^2}{4}}d\gamma(x)\leq\frac{1}{\gamma(A)}\leq 2$$

and consequently we have $\gamma(\overline{A_t})\leq 2e^{-\frac{t^2}{4}}$, as required.

□

Remark 14. Infact, an even stronger concentration inequality holds, we have in the notation of Theorem 12, $\gamma(\overline{A_t})\leq e^{-\frac{t^2}{4}}$.

Remark 15. A similar result holds for the sphere S^n with uniform measure μ, if A is a subset with $\mu(A)\geq\frac{1}{2}$, then $\mu(\overline{A_t})\leq 2e^{-\frac{t^2}{2n}}$.

Let us consider an alternative viewpoint. Suppose A is a subset of S^n, with $\mu(A)=\frac{1}{2}$. Maximizing (over all A with $\mu(A)=\frac{1}{2}$) $\mu(\overline{A_t})$ is the same thing as minimizing $\mu(A_t\backslash A)$. Since we have $\lim_{t\to 0}\frac{\mu(A_t\backslash A)}{t}=\mu_{n-1}(\partial A)$, an equivalent question is for which A with $\mu(A)=\frac{1}{2}$ is $\mu_{n-1}(\partial A)$ minimum? More generally, we can ask the question, for any $p\in(0,1)$, for which subset A of the sphere S^n with $\mu(A)=p$ is $\mu_{n-1}(\partial A)$ minimum? The following theorem shows that caps are a solution to the above question.

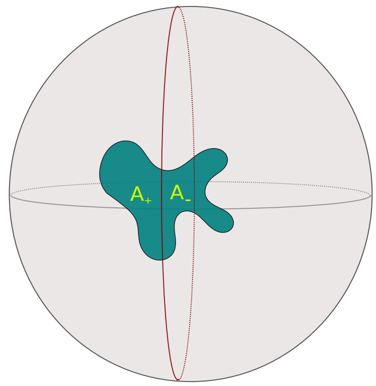

FIGURE 14: A hyperplane bisecting A

Theorem 16. *A cap (intersection of the sphere with a half-space) of the sphere with measure $p \in (0, 1)$ has minimum boundary measure among all subsets of measure p.*

Proof. Suppose A is a subset of measure p with minimum boundary measure. Consider any hyperplane that bisects A (i.e. divides A into two parts A_+ and A_- with $\mu(A_+) = \mu(A_-)$).

By reflecting A_+ accross this hyperplane we can symmetrize A along this hyperplane. If we symmetrize with respect to every possible great circle bisecting A, we are left with a cap. Hence the result. □

Remark 17. Similar result holds for $\mathbb{R}^n$ with Gaussian density γ, but not the unit ball. For example, the following example shows that a ball and a cap of same area inside the unit ball in the plane, where the ball has smaller perimeter than the cap.

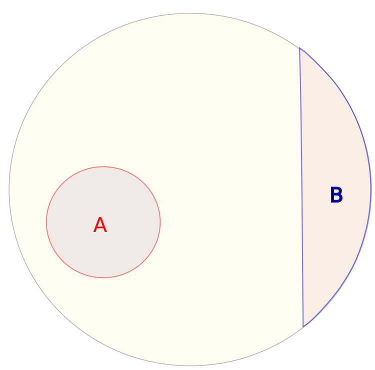

FIGURE 15: In the unit ball in the plane, the regions A and B have the same area, but A has less perimeter than B, although B is cut off by a hyperplane

One can ask similar question with density defined by an arbitrary logconcave function instead of the Gaussian. Let $f : \mathbb{R}^n \to [0, \infty)$ be a logconcave function with $\int_{\mathbb{R}^n} f < \infty$, and let π_f be the associated measure (i.e. $\pi_f(S) = \int_S f$).

Theorem 18 (KLS Theorem). *For any logconcave density f and any partition $\mathbb{R}^n = S_1 \cup S_2 \cup S_3$, we have*

$$\pi_f(S_3) \geq \frac{\ln 2d(S_1, S_2)}{\mathbb{E}_f \|x - \bar{x}\|^2} \min\{\pi_f(S_1), \pi_f(S_2)\}.$$

We see that we get analogous results to Theorem 12 if we take $S_1 = A$, $S_2 = A_t \setminus A$.

Remark 19. With the same notation as Theorem 18, a slightly different inequality is

$$\pi_f(S_3) \geq \frac{2d(S_1, S_2)}{\text{Diam (Support}(f))} \min\{\pi_f(S_1), \pi_f(S_2)\}.$$

Suppose f is a logconcave density function with $\int_{\mathbb{R}^n} f(x)dx = 1$, and the center of gravity $\int_{\mathbb{R}^n} xf(x)dx$ is the origin. We can consider a halfspace H cut out by a hyperplane passing through the origin. If f is centrally symmetric, then for any such halfspace H, we have $\pi_f(H) = \frac{1}{2}$. For an arbitrary logconcave density f, we would like to have bounds on $\pi_f(H)$.

Example 20 (Simplex and Cone). For the standard n-simplex $\Delta^n \subseteq \mathbb{R}^n$, if we take $f = \mathbb{1}_{\Delta^n}$, then for the hyperplane H parallel to one of the faces of Δ^n we see that one halfspace H_n contains a dilated (by a factor of $\frac{n}{n+1}$) copy of the standard Δ^n, and so we have $\pi_f(H_n) = (\frac{n}{n+1})^n$. We see that asymptotically $\pi_f(H_n) \downarrow \frac{1}{e}$.

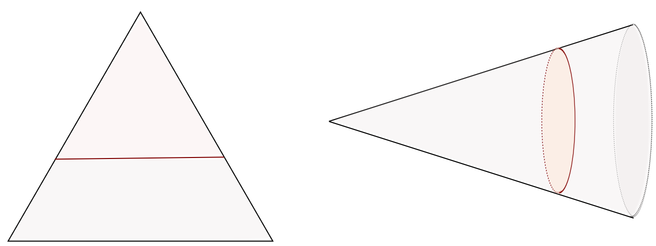

Figure 16: A 2-simplex and a cone bisected by hyperplanes

The exact same results hold for the standard cone.

The following theorem shows that $\frac{1}{e}$ is a lower bound for any such f, and the above example shows that we cannot get better bounds.

Theorem 21 (Grunbaum). *For any logconcave density f and any halfspace H containing the center of gravity, we have*

$$\frac{1}{e} \leq \pi_f(H) \leq 1 - \frac{1}{e}.$$

We will prove this result in the special case f is indicator function of a convex body of unit volume.

Proof. We observe that it suffices to prove the lower bound, since the upperbound follows from the lower bound of the complementary halfspace. Let $f = \mathbb{1}_S$ where S is a convex body. Let us consider a hyperplane ℓ cutting out halfspace H containing the center of gravity.

We will modify S a number of times to get a cone, each time making sure that the volume of the new halfspace is same as the volume of the old one and the new center of gravity remains in the new halfspace, and we will use the volume of the cone to show the lower bound.

Firstly, let us replace sections with balls of the same volume. As we saw in the proof of Theorem 2, this symmetrization S_1 is convex, and the center of gravity remains the same.

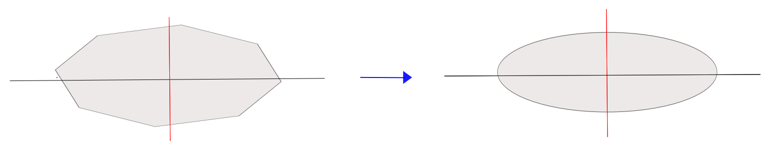

FIGURE 17: Symmetrizing S to S_1

Let us pick a cone of the same volume as $H \cap S_1$ (left side in the figure) and with base $\ell \cap S_1$, and we call the resulting body S_2. We observe that the center of gravity remains in H (possibly moves to the left, i.e. away form ℓ).

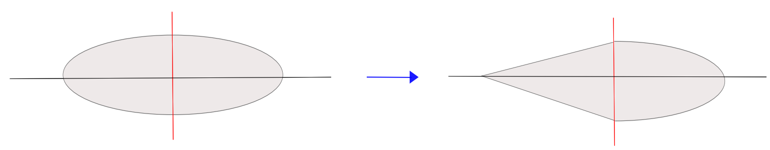

FIGURE 18: Replacing $H \cap S_1$ with a cone to get S_2

Finally, let us replace $\overline{H} \cap S_2$ with a body the same volume, and so that the resulting body C is a standard cone. Once again, we see that the center of gravity remains in H (possibly moves to the left, i.e. away form ℓ).

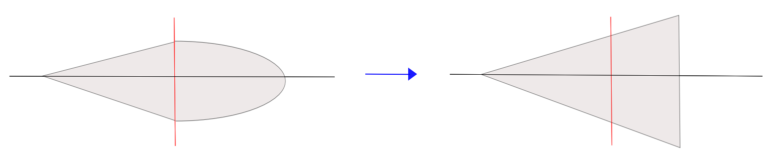

FIGURE 19: Replacing $\overline{H} \cap S_2$ to get a cone C

Now $H \cap C$ contains a smaller cone dilated by a factor of $\frac{n}{n+1}$, and thus by Example 20, we have volume of $H \cap C$ is atleast $\frac{1}{e}$, and thus $\pi_f(H) \geq \pi_{\mathbb{1}_C}(H) \geq \frac{1}{e}$, as required. $\square$